Maxmin on 3-sphere.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Among all triples of real numbers (x, y, z) such that $x^2 + y^2 + z^2 = 1$ find the triple which maximize $\min\{|x - y|, |y - z|, |z - x|\}$.

Solution.

Due to symmetry of the function $\min\{|x - y|, |y - z|, |z - x|\}$ we can consider only triples (x, y, z) such that $x^2 + y^2 + z^2 = 1$ and $x \le y \le z$.

In that case any such triple can be represented in form $(x, x + p, x + p + q)$ where p, q are any non-negative real and x is real numbers such that

(1) $\quad x^2 + (x + p)^2 + (x + p + q)^2 = 1$

and $\min\{|x - y|, |y - z|, |z - x|\} = \min\{p, q, p + q\} = \min\{p, q\}$.

Thus we need maximize $\min\{p, q\}$, where $x, p, q \in \mathbb{R}, p, q \ge 0$ and (1)$\Leftrightarrow$

(2) $\quad 3x^2 + 2x(2p + q) + 2p^2 + 2pq + q^2 - 1 = 0.$

Let $D = (2p + q)^2 - 3(2p^2 + 2pq + q^2 - 1)$ is discriminant of quadratic equation (2).

Since equation (2) solvable in $\mathbb{R}$ iff $D \ge 0 \Leftrightarrow$

$4p^2 + 4pq + q^2 - 6p^2 - 6pq - 3q^2 + 3 \ge 0 \Leftrightarrow 3 \ge 2p^2 + 2q^2 + 2pq$

then our problem is find $\max\limits_{p,q}(\min\{p, q\})$, where $p, q \ge 0$ and $2p^2 + 2q^2 + 2pq \le 3$.

Let $t = \min\{p, q\} \ge 0$ then $p, q \ge t$ and, therefore, $3 \ge 2p^2 + 2q^2 + 2pq \ge 6t^2 \Leftrightarrow$ $t^2 \le \frac{1}{2} \Leftrightarrow t \le \frac{1}{\sqrt{2}}$.

Thus, $\min\{p, q\} \le \frac{1}{\sqrt{2}}$ and this upper bound can be attained if we take $p = q = \frac{1}{\sqrt{2}}$ and then $x = -\frac{2p + q}{3} = -\frac{1}{\sqrt{2}}$ is only root of equation (2).

So, $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ maximize $\min\{|x - y|, |y - z|, |z - x|\}$ and this maximum equal $\frac{1}{\sqrt{2}}$.